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重力波の非線型変調

東大　宇宙研　橋本英典
京大　理　　　小野広明

§1　序

一定深さの水に生じる、伝播波は波の分散と非線型性のバランスによって維持されるものであって、ストークスによる古典的な研究に始まり、多くの人々によってそのくわしい形を求める研究が今日に至るまで行なわれて来た。従ってこのような波がその変調に対して不安定なことがライトヒル(1965, 1967)、ウィザム(1967)、ベンジャミンとフェア(1967)等によって、その結論のあるものが確認されたことは流体力学における劇的な発見の1つといえよう。しかし、以上の理論は初期の段階において、爆発的な不安定がおこることを示すだけで、そのような不安定がどのように発展して行くかについては何も教えてくれない。

最近、このような不安定が振幅が時空的にゆっくり変化するいわゆる変調波として存在することを前提とする解析がチューとメイ(1970)によって試みられている。彼等はWKB法に似た

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解析により、ウィザムの方法による方程式の中に変調の変化率に相当する新しい項が加わることを示した。しかしその得た方程式系は非常に複雑で、具体的な結果は無限に深いばあいを、更に変調が小さいという假定のもとにとりあつかっているに過ぎない(1971)。

こゝでは水の深さh_0が有限とした解析が行なわれる。無限に広がった媒質のばあいにくらべ、こゝでは深さ方向の変数が問題を複雑にするが、演算子に対する特異摂動法の採用によってこれも解決することができる。その結果は一様媒質に対する座標のひきのばしによるフーリエ展開法を拡張した方法による結果と一致することが示される。

得られた波の発展を記述する方程式は、最低次において非線形のシュレーディンガー方程式であり、プラズマ物理、非線型光学、液体ヘリウムの理論、また流体力学においては非常に細い渦糸の運動の記述にあらわれることが知られている。

この方程式の著しい特長は、1) Stokes の波列がこの方程式の平面波解に対応する、2) ベンジャミン(1967)やウィザム(1967)によって示されたように波長とh_0の比が一定の臨界値を超すと$h_0 \to \infty$での変調不安定が消失することが示される。3) 浅水波の極限ではコルトベーグ、ドブリエ方程式によって支配される弱い波列解が導かれるなどの重要な効果が含まれる点である。

§2. Fundamental equations

The wave propagation on water of uniform depth h_0 is governed by the Laplace equation

$$\Delta \Phi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = 0, \tag{2.1}$$

for the velocity potential $\Phi(x, y, t)$ subject to the boundary conditions

$$\Phi_y = 0 \qquad \text{at} \quad y = -h_0, \tag{2.2}$$

$$\Phi_y = \eta_t + \Phi_x \eta_x \qquad \text{at} \quad y = \eta(x,t), \tag{2.3}$$

and

$$\Phi_t + g\eta + \frac{1}{2} v^2 = 0 \qquad \text{at} \quad y = \eta(x,t), \tag{2.4}$$

or

$$\Phi_{tt} + g\Phi_y + \left[\frac{\partial}{\partial t} + \frac{1}{2}(\mathbf{V}\cdot\nabla)\right]\mathbf{V}^2 = 0 \quad \text{at} \quad y = \eta(x,t), \tag{2.4a}$$

after elimination of $\eta(x,t)$.

Here, y = 0 is the unperturbed and y = η (x,t) the perturbed surface, g the gravity and the velocity $\mathbf{V}$ is derived from Φ as

$$\mathbf{V} = \nabla \Phi . \tag{2.5}$$

Let us introduce the operators ϵ and p:

$$\epsilon = i\frac{\partial}{\partial t} \qquad \text{and} \qquad p = -i\frac{\partial}{\partial x} . \tag{2.6}$$

Then, the solution of (2.1) satisfying the boundary condition (2.2) is given in operational form as follows :

$$\Phi = \frac{\cosh((y+h_0)p)}{\cosh h_0 p}[f(x,t)] = \{1 + ypT + \tfrac{1}{2}y^2p^2 + \tfrac{1}{6}y^3p^3T + O(y^4)\}[f], \tag{2.7}$$

where

$$T = \tanh h_0 p , \tag{2.8}$$

and f (x, t) is the value of Φ at y = 0.

Introducing (2.6) into (2.4a) and (2.4), and retaining quantities upto the third order terms in η and f which are assumed to be proportianal to a small parameter ε, we have

$$L[f] \equiv (\epsilon^2 - g p T)[f] = \eta \Lambda[f] - \tfrac{1}{2}\eta^2 L[p^2 f] + i(\epsilon Q - 2\eta\epsilon G) + H + O(\varepsilon^4), \qquad (2.9)$$

where

$$\Lambda = g p^2 - \epsilon^2 p T, \qquad (2.10)$$

$$Q = (pf)^2 - (pTf)^2, \qquad (2.11)$$

$$G = (p^2 f)(pTf) - (pf)(p^2 T f), \qquad (2.12)$$

$$H = (pf)\{(pf)(p^2 f) - (pTf)(p^2 T f)\} + (pTf)G, \qquad (2.13)$$

and

$$g\eta = i\epsilon f - \frac{1}{g}(\epsilon f)(\epsilon pTf) + \frac{1}{2}Q + O(\varepsilon^3). \qquad (2.14)$$

§3. Perturbation scheme

In order to investigate uniformly valid perturbation solutions of (2.9) and (2.14) it is convenient to introduce variables of multiple scales

$$t, \quad t_1 = \varepsilon t, \quad \tau = \varepsilon^2 t; \qquad (3.1)$$

$$x, \quad x_1 = \varepsilon x, \qquad (3.2)$$

and expand f and η in series of the form :

$$f = \sum_{n=1} \varepsilon^n f_n(x, x_1; t, t_1, \tau), \tag{3.3}$$

$$\eta = \sum_{n=1} \varepsilon^n \eta_n(x, x_1; t, t_1, \tau). \tag{3.4}$$

Introducing (3.1) - (3.4) into (2.9) and (2.14) and equating like powers of ε, we find

$$L[f_1] = 0, \tag{3.5}$$

$$L[f_2] = -L_1[f_1] + \eta_1 \Lambda[f_1] + i\epsilon[(pf_1)^2 - (pIf_1)^2], \tag{3.6}$$

$$\begin{aligned} L[f_3] = & -L_1[f_2] - L_2[f_1] + \eta_1 \Lambda[f_2] + \eta_2 \Lambda[f_1] \\ & + \eta_1 p_1 \Lambda'[f_2] - 2\eta_1 \epsilon_1 \epsilon\, pI[f_1] + i\epsilon_1 Q \\ & + 2i\epsilon[(pf_1)(pf_2) - (pIf_1)(pIf_2) + (pf_1)(p_1 f_1) - (pIf_1)(pI)'[p_1 f_1]] \\ & + H - 2i\eta_1 \epsilon G, \end{aligned} \tag{3.7}$$

and

$$g\eta_1 = i\epsilon f_1, \tag{3.8}$$

$$g\eta_2 = i\epsilon f_2 - \frac{1}{g}(\epsilon f_1)(pIf_1) + \frac{1}{2}Q, \tag{3.9}$$

where

$$\epsilon_1 = i\frac{\partial}{\partial t_1}, \quad \epsilon_2 = i\frac{\partial}{\partial \tau}, \quad p_1 = \frac{1}{i}\frac{\partial}{\partial x_1}, \tag{3.10}$$

$$L_1 \equiv L_\epsilon \epsilon_1 + L' p_1 = 2\epsilon\epsilon_1 - (gpI)' p_1, \tag{3.11}$$

$$\begin{aligned} L_2 \equiv & L_\epsilon \epsilon_2 + \frac{1}{2}L_{\epsilon\epsilon}\epsilon_1^2 + \frac{1}{2}L'' p_1^2 + L'_\epsilon \epsilon_1 p_1, \\ & (= 2\epsilon\epsilon_2 + \epsilon_1^2 - \frac{1}{2}(gpI)'' p_1^2) \end{aligned} \tag{3.12}$$

and the prime denotes the derivative with respect to p.

If we let Z and $\bar{Z}$ denote

$$Z = \exp i\,(k_0 x - \omega_0 t)\,, \tag{3.13}$$

and its complex conjugate (C. C.), we have

$$P(\epsilon, p)\,[Z^n] = P(n\omega_0, n k_0)\,Z^n, \tag{3.14}$$

and

$$P(\epsilon, p)\,[\bar{Z}^n] = P(-n\omega_0, -n k_0)\,\bar{Z}^n. \tag{3.15}$$

Then, the solution of (3.5) for a progressive wave is given by

$$f_1 = \psi Z + \bar{\psi}\bar{Z} + \varphi, \tag{3.16}$$

provided that the dispersion relation

$$L[\omega_0, k_0] = 0\,, \quad \text{i.e. } \omega_0^2(k_0) = g k_0 \sigma, \tag{3.17}$$

with

$$\sigma = \tanh k_0 h_0, \tag{3.18}$$

is satisfied, where ψ and φ are functions of slow variables x_1 , t_1 and τ , and k_0 and ω_0 are respectively the wave number and the frequency of infinitesimal wave.

From (3.16) and (3.8) we have

$$g\eta_1 = i\omega_0(\psi Z - C.C.). \tag{3.19}$$

Introduction of (3.16) and (3.19) into (3.6) gives

$$L[f_2] = -L_1[f_1] + 3i\omega_0 k_0^2\,(1-\sigma^2)(\psi^2 Z^2 - C.C.) \tag{3.20}$$

which yields a uniformly valid solution

$$f_2 = \frac{3i(1-\sigma^4)}{4\sigma^2\omega_0} k_0^2 \psi^2 Z^2 + \text{C. C.}, \tag{3.21}$$

provided that the secular term

$$\begin{aligned} L_1[\psi Z] &= L_\epsilon[Z] \left(\frac{\partial}{\partial t_1} + V\frac{\partial}{\partial x_1}\right)\psi \\ &= 2i\,\omega_0 Z \left(\frac{\partial}{\partial t_1} + V\frac{\partial}{\partial x_1}\right)\psi, \end{aligned} \tag{3.22}$$

and its C. C. vanish everywhere, where

$$\begin{aligned} V(k_0) &= -L_{k_0}(\omega_0, k_0)/L_{\omega_0}(\omega_0, k_0) = \omega_0'(k_0) \\ &= \frac{1}{2} c_0 \{1 + (1-\sigma^2) k_0 h_0/\sigma\}, \end{aligned} \tag{3.23}$$

and

$$c_0 = \omega_0/k_0 = (g\sigma/k_0)^{\frac{1}{2}}, \tag{3.24}$$

are respectively the group and phase velocity for infinitesimal wave, and the secular term $L_1[\varphi]$ is found to be zero. This condition gives

$$\psi = \psi(\xi, \tau), \tag{3.25}$$

where the variable

$$\xi = x_1 - V t_1, \tag{3.26}$$

shows that the slow modulation of the wave due to weak non-linearity is propagated approximately with the group velocity V.*)

Introducing (3.16), (3.21) and (3.17) with (3.25) into (3.9) we have

$$\begin{aligned} g\eta_2 &= \gamma_2 \psi^2 Z^2 + V\psi_\xi Z + \text{C. C.} \\ &\quad - (1-\sigma^2) k_0^2 \psi\bar{\psi} - \partial\varphi/\partial t_1, \end{aligned} \tag{3.27}$$

where

$$\gamma_2 = \frac{1}{2}(\sigma^2 - 3) k_0^2/\sigma^2, \tag{3.28}$$

*) The choice of the order of the coordinate stretching such as (3.25), and (3.26)[8-10] is uniquely determined also by taking into account the balance between the dispersion and nonlinear effects[8,14].

§ 4. Non-linear Schrödinger equation

In order to find the equation governing $\psi(\xi, \tau)$ in § 3 we must proceed to the third order equation (3.7) and impose the condition that the right hand side contains no secular terms Z, Z^0 and $\bar{Z}$. After straightforward algebra and making use of the formulae

$$(gpT)''[Z^0] = \{\omega_0^2(k_0)\}''_{k_0=0} \doteqdot 2gh_0,$$
$$V' L_{\omega_0} + V^2 L_{\omega_0\omega_0} + 2V L_{\omega_0 k_0} + L_{k_0 k_0} = 0, \tag{4.1}$$

we have from the coefficient of Z^0

$$-L_2[\varphi] \equiv \left(\frac{\partial^2}{\partial t_1^2} - g h_0 \frac{\partial^2}{\partial x_1^2}\right)\varphi = \{2\omega_0 k_0 + (1-\sigma^2)k_0^2 V\}(\psi\bar{\psi})_\xi \tag{4.2}$$

and from that of Z (and $\bar{Z}$)

$$-Z^{-1}L_2[\psi Z] \doteqdot 2\omega_0\left(i\frac{\partial\psi}{\partial\tau} + \frac{1}{2}V'\psi_{\xi\xi}\right)$$
$$= [2\omega_0 k_0 \partial\varphi/\partial x_1 - (1-\sigma^2)k_0^2 \partial\varphi/\partial t_1]\psi \tag{4.3}$$
$$+ [-(1-\sigma^2)^2 + \frac{1}{2\sigma^2}(9 - 10\sigma^2 + 9\sigma^4)]k_0^4 \psi\psi\bar{\psi}.$$

Equation (4.2) is integrated on the assumption that φ is a function of ξ and τ and yields the induced horizontal current due to nonlinear interaction :

$$\varphi_\xi = [2\omega_0 k_0 + (1-\sigma^2)k_0^2 V]\,\psi\bar{\psi}/(V^2 - gh_0). \tag{4.4}$$

Introducing (4.4) into (4.3) and rearranging we obtain

$$\frac{1}{i}\frac{\partial\psi}{\partial\tau} = \mu\frac{\partial^2\psi}{\partial\xi^2} + \nu|\psi|^2\psi, \tag{4.5}$$

where

$$\mu = \frac{1}{2} V_0'(k_0) = \frac{1}{2} \omega_0''(k_0) = -\frac{g}{8 k_0 \sigma \omega_0}\left[\{\sigma - k_0 h_0 (1-\sigma^2)\}^2 + 4 k_0 h_0^2 (1-\sigma^2)\right], \tag{4.6}$$

and

$$\nu = \frac{-k_0^4}{2\omega_0}\left[\frac{1}{V^2 - g h_0}\{4C_0^2 + 4(1-\sigma^2) C_0 V + g h_0 (1-\sigma^2)^2\} + \frac{1}{2\sigma^2}(9 - 10\sigma^2 + 9\sigma^4)\}\right]. \tag{4.7}$$

As is seen from (4.6), μ takes always negative sign, whereas ν changes its sign from negative to positive across $k_0 h_0 = 1.363$ as $k_0 h_0$ decreases. It should be noted that $-\nu$ is essentially identical with $X(K)$ defined by the equation (30) in Benjamin's paper[5]. An equation of this type, which may be called a nonlinear Schrödinger equation, has already been obtained for various problems [9-13]. A generalized equation, in which both μ and ν are complex, has also been obtained by Stewartson and Stuart[15] in the study of the nonlinear instability of plane Poiseuille flow.

If, instead of the complex amplitude $\psi(\xi, \tau)$, we use the pair of real functions A and Ω defined by :

$$\psi = A \exp\left[\frac{i}{2\mu}\int \Omega \, d\xi\right], \tag{4.8}$$

then we obtain the following set of equations :

$$\frac{\partial A^2}{\partial \tau} + \frac{\partial}{\partial \xi}(A^2 \Omega) = 0, \tag{4.9}$$

$$\frac{\partial \Omega}{\partial \tau} + \Omega \frac{\partial \Omega}{\partial \xi} - 2\mu\nu \frac{\partial A^2}{\partial \xi} - 2\mu^2 \frac{\partial}{\partial \xi}\left(\frac{1}{A}\frac{\partial^2 A}{\partial \xi^2}\right) = 0, \tag{4.10}$$

which reduces, in the limit of $k_0 h_0 \to \infty$, to the set of equations of the form adopted by Chu and Mei[7].

The elevation η is determined in terms of ψ from (3.19) , (3.27) and (4.4) as

$$g\eta = i\varepsilon\omega_0 \psi Z + \varepsilon^2 (V\psi_\xi Z + \gamma_2 \psi^2 Z^2) + c.c. \qquad (4.11)$$
$$+ \varepsilon^2 \gamma \psi \overline{\psi},$$

where

$$\gamma = \frac{1}{V^2 - gh_0}\{2\omega_0 k_0 V + (1-\alpha^2) g h_0 k_0^2\}. \qquad (4.12)$$

§5. Several solutions of the nonlinear Schrödinger equation (4.5)

5 - 1 Nonlinear plane wave solution

It is known that (4.5) has the following solution representing a nonlinear plane wave :

$$\psi = \psi_0 \exp\{i(\alpha\tau - \kappa\xi)\}, \qquad (5.1)$$

where

$$\psi_0 = constant, \quad \alpha = -\mu\kappa^2 + \nu|\psi_0|^2. \qquad (5.2)$$

Let us now consider the meaning of this solution in the original physical variables. In particular, if we set $\kappa = 0$ and $\psi_0 = ga/(2i\omega_0)$, a being a real constant, then the perturbed surface given by (4.11) takes the following form :

$$\eta = \varepsilon a \cos\zeta + \frac{1}{4}(\varepsilon^2 a^2/\sigma k_0)(\gamma - \gamma_2 \cos 2\zeta), \tag{5.3}$$

where

$$\zeta = k_0 x - (\omega_0 - \varepsilon^2 \alpha_0) t, \text{ with } \alpha_0 = \frac{1}{4}\nu g^2 a^2/\omega_0^2. \tag{5.4}$$

This is nothing but the Stokes wave train to the second order approximation. Here, it should be noted that $\omega = \omega_0 - \varepsilon^2 \alpha_0$ is the nonlinear dispersion relation for Stokes wave including the effect of the mean horizontal current. It is also to be noted that the dispersion term in (4.5) plays no essential role in this solution because $\kappa = 0$.

5 - 2 Equilibrium solution

In addition to the plane wave solution described above, eq. (4.5) has another type of solution in terms of the Jacobian elliptic function, exhibiting the dynamical balance between nonlinear and dispersion effects, which we call equilibrium solution :

$$\psi = A(\xi) \exp(i\alpha\tau), \tag{5.5}$$

where

$$\alpha \text{ is constant and } A \text{ is real}^{*)} \tag{5.6}$$

a) In the case of $\mu\nu > 0$.

$$A(\xi) = A_0 \, dn\{A_0(\tfrac{1}{2}\nu/\mu)^{\frac{1}{2}}\xi, \, s\}, \tag{5.7}$$

with the modulus s

$$s^2 = 2 - 2\alpha/(\nu A_0^2). \tag{5.8}$$

In the special case of $s = 1$ we have eq. (6.8) (to be mentioned in § 6.

b) In the case of $\mu\nu < 0$.

$$A(\xi) = A_0 \, sn\{(-\tfrac{1}{2}\nu/\mu)^{\frac{1}{2}} A_0 \xi/s,\ s\}, \tag{5.9}$$

with the modulus

$$s^2 = A_0^2 / (2\alpha/\nu - A_0^2). \tag{5.10}$$

In the special case of $s = 1$, we have

$$A(\xi) = (\alpha/\nu)^{\frac{1}{2}} \tanh\{(-\tfrac{1}{2}\alpha/\mu)^{\frac{1}{2}} \xi\}, \tag{5.11}$$

which describes the propagation of a phase jump.

*) If a complex form of A is permitted, we obtain an equilibrium solution of slightly generalized type. For the aim of later discussions, however, this simple choice may be sufficient.

§6. Stability of the Stokes wave (5.3)

The stability of the Stokes wave has been investigated by sereral authors[1-5] both analytically and experimentally. We shall show that the time evolution of the unstable modes may be regarded as a special case of the general modulation processes described by (4.5)

In order to reproduce the Stokes wave, let us set $\alpha = \alpha_0$, $k = 0$ and $\psi_0 = ga/(2i\omega_0)$ in (5.1). Then we consider a disturbed Stokes wave given by

$$\psi = (\psi_0 + \hat{\varepsilon}\hat{\psi}) \exp i(\alpha_0 \tau + \hat{\varepsilon}\hat{\theta}), \tag{6.1}$$

where $\hat{\psi}$ and θ are assumed to be real functions representing the disturbance, $\hat{\varepsilon}$ being a small parameter. Substituting the above expression into (4.5) and linearizing it with respect to $\hat{\varepsilon}$, we have

$$\hat{\psi}_\tau + \mu |\psi_0| \hat{\theta}_{\xi\xi} = 0, \tag{6.2}$$

$$\hat{\theta}_\tau - 2\nu |\psi_0|^2 \hat{\psi} - \mu \hat{\psi}_{\xi\xi} = 0. \tag{6.3}$$

Since these equations form a set of linear differential equations with constant coefficients, we can assume a solution of the form :

$$\begin{pmatrix} \hat{\psi} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \hat{\psi}_0 \\ \hat{\theta}_0 \end{pmatrix} e^{i(\hat{k}\xi - \hat{\omega}\tau)} + C.C., \tag{6.4}$$

where $\hat{\psi}_0$ and $\hat{\theta}_0$ are constant. From the condition that (6.2) have a non-trivial solution, we obtain the dispersion relation :

$$\hat{\omega}^2 = \mu^2 \hat{k}^2 (\hat{k}^2 - 2\nu |\hat{\psi}_0|^2/\mu), \tag{6.5}$$

which shows that, if $\mu\nu < 0$, $\hat{\omega}$ is always real for arbitrary values of $\hat{k}$ so that the Stokes wave given by (5.3) is neutrally stable. On the other hand, if $\mu\nu > 0$, $\hat{\omega}$ becomes imaginary for

$$\hat{k} < \sqrt{2\nu/\mu}\,|\hat{\varphi}_0| \tag{6.6}$$

Hence the disturbance will grow exponentially. In this sense, the Stokes wave given by (5.3) is unstable against the above modulational disturbance, and the maximum growth rate, say δ_{max}, is given by

$$\delta_{max} = |\nu\hat{\varphi}_0^2| \quad \text{for} \quad \hat{k} = (\nu/\mu)^{\frac{1}{2}}|\hat{\varphi}_0| \tag{6.7}$$

Remembering the discussion concerning the signs of μ and ν given in § 4, we may conclude that these results reproduce those obtained by Benjamin[5] and Whitham[3]. In the present theory, returning to the original nonlinear Schrödinger equation (4.5), we can investigate further time evolution of such unstable modes even to the stage when the linear theory fails to be valid. For example, when $s = 1$ the equilibrium solution (5.7) degenerates into a solitary modulational wave propagating with the group velocity;

$$A(\xi) = (2\alpha/\nu)^{\frac{1}{2}}\,\mathrm{sech}\{(\alpha/\mu)^{\frac{1}{2}}\xi\} \tag{6.8}$$

which has the width $(\mu/\alpha\varepsilon^2)^{\frac{1}{2}}$. This width, when $\alpha = \alpha_0$, agrees with the wave length of the unstable mode with maximum growth rate. This fact leads us to a conjecture that the modulation of the Stokes wave is eventually deformed into the solitary wave described by (6.8). In fact, the numerical calculations carried out by Chu and Mei[7] Karpman and Kruskhal[9] and by Yajima and Outi[16], strongly support this conjecture.

§7. The nonlinear Schrödinger equation (4.5) in the shallow-water limit

In order to show a wide applicability of eq. (4.5), we shall discuss the equation in the shallow-water limit. In the limit of $k_0 h_0 \to 0$ with k_0 kept to be of the order of unity, the coefficients μ and ν in eq. (4.5) become, respectively, as follows :

$$\mu_s = -\frac{1}{2} c_0^{\frac{1}{2}} k_0 h_0^2, \quad \nu_s = \frac{9}{4} c_0^{-\frac{1}{2}} k_0 h_0^{-2}, \tag{7.1}$$

where

$$c_0 = (g h_0)^{\frac{1}{2}}, \tag{7.2}$$

whence the nonlinear plane wave given by (5.3) assumes the following form for $\varepsilon < (k_0 h_0)^3 \ll 1$

$$\eta = \varepsilon a \cos \zeta_s - \frac{3}{4} \varepsilon^2 a^2 k_0^{-2} h_0^{-3} (1 - \cos 2\zeta_s) + O(\varepsilon^3), \tag{7.3}$$

where

$$\zeta_s = k_0 x - (\omega_0 - \varepsilon^2 \alpha_s) t, \tag{7.4}$$

On the other hand, as is well known, the shallow-water waves are governed by the Korteweg-de Vries equation [17] :

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{3}{2} \frac{c_0}{h_0} \eta \frac{\partial \eta}{\partial x} + \frac{h_0^2}{6} c_0 \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{7.5}$$

which has the steady periodic solution called cnoidal wave :

$$\eta = \varepsilon a [\eta_\infty + 2 s^{-2} dn^2 \{ (\varepsilon a / 6 s)^{\frac{1}{2}} (x - Vt), s \}], \tag{7.6}$$

where

$$V = c_0 [1 + \frac{3 \varepsilon a}{2 h_0} \{ \eta_\infty + \frac{2}{3} (\frac{2}{s^2} - 1) \}], \tag{7.7}$$

and the mean depth, say $\bar{\eta}$, is given by

$$\bar{\eta} = \varepsilon a [\eta_\infty + 2 s^{-2} E/K], \tag{7.8}$$

where K , E , and s are respectively, the first and the second kind of complete integrals and their modulus. Putting

$$k_0^2 = \frac{3}{2}\varepsilon a\pi^2/(s^2 h_0^3 K^2) \text{ and } \bar{\eta} = -\frac{3}{4}\varepsilon^2 a^2/(h_0^3 k_0^2), \quad (7.9)$$

and expanding (7.8) for small values of s , we have (7.3). Thus we find that the nonlinear plane wave solution corresponds to the weak cnoidal wave in the shallow-water limit. We note here that we can obtain the nonlinear Schrödinger equation (4.5) with $\mu = \mu_s$ and $\nu = \nu_s$ directly from the Korteweg-de Vries equation (7.5) by the same procedure adopted in previous sections. Similar procedure was also adopted by Tappert and Varma[12] in the study of heat pulses in solids. According to the criterion of the stability discussed in § 6 , we may conclude that the weak cnoidal wave is modulationally stable against the small disturbance (as shown numerically by Zabusky and Kruskal[15]), because $\mu_s \nu_s < 0$.
For the complementary case to the weak cnoidal wave considered here , Jeffrey and Kakutani[17] showed, by the conventional stability theory, that the solitary wave solution of the Korteweg-de Vries equation is neutarlly stable. Berezin and Karpman[20] also investigated an asymptotic behaviour of the cnoidal wave for arbitrary values of the modulus s by starting from a formulation due to Whitham who did not take a dispersion term (the last term in eq. (4.10)) into account as was remarked in § 1.

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Appendix A. Fourier series expansion

By use of the stretched coordinates $\xi = \varepsilon(x - Vt)$ and $\tau = \varepsilon^2 t$ introduced by Taniuti and Yajima[10], we expand Φ and η into series of the form

$$\Phi(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \phi^{(n,m)}(\xi,y,\tau)\,Z^{m} \tag{A.1}$$

$$\eta(x,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \eta^{(n,m)}(\xi,\tau)\,Z^{m} \tag{A.2}$$

where

$$\phi^{(n,-m)} = \bar{\phi}^{(n,m)} \quad \text{and} \quad \eta^{(n,-m)} = \bar{\eta}^{(n,m)} \tag{A.3}$$

since Φ and η are real.

Substituting (A.2) (A.3) into (2.1) - (2.4a) and separating different order of ε and harmonics, we have a set of ordinary linear differential equations for $\phi^{(n,m)}$ and $\eta^{(n,m)}$:

$$\phi_{yy}^{(n,m)} - m^2 k_0^2 \phi^{(n,m)} = A^{(n,m)}(\xi,y,\tau) \qquad \text{for } -h_0 \leqq y \leqq 0 \tag{A.4}$$

$$\phi_y^{(n,m)} = 0 \qquad \text{at} \quad y = -h_0 \tag{A.5}$$

$$\phi_y^{(n,m)} + i m w_0 \eta^{(n,m)} = B^{(n,m)}(\xi,\tau) \qquad \text{at} \quad y = 0 \tag{A.6}$$

and

$$-i m \omega_0 \phi^{(n,m)} + g\,\eta^{(n,m)} = C^{(n,m)}(\xi,\tau) \qquad \text{at} \quad y = 0 \tag{A.7}$$

or after elimination of η

$$g\,\phi_y^{(n,m)} - m^2\omega_0^2 \phi^{(n,m)} = g B^{(n,m)} - i m \omega_0 C^{(n,m)} \qquad \text{at } y = 0 \tag{A.8}$$

where $A^{(n,m)}$, $B^{(n,m)}$ and $C^{(n,m)}$ contain the lower order quantities with respect to n, (their explicit forms are given in Appendix B) and we have replaced the boundary conditions (2.3) and (2.4) at $y = \eta(x,t)$ by those at $y = 0$ by use of powers series in η :

$$\Phi_y - \eta_t = -\sum_{N=1}^{\infty} \frac{\eta^N}{N!}\left(\frac{\partial^{N+1}\Phi}{\partial y^{N+1}} - \eta_x \frac{\partial^{N+1}\Phi}{\partial x \partial y^N}\right), \quad (2.3A)$$

and

$$\Phi_{tt} + g\Phi_y = -\sum_{N=1}^{\infty} \frac{\eta^N}{N!}\left[\frac{\partial^N}{\partial y^N}(\Phi_{tt} + g\Phi_y) - \frac{1}{2}\sum_{M=0}^{\infty}\left\{\binom{N}{M}\left(\frac{\partial^M \Phi_x}{\partial y^M}\frac{\partial^{N-M}\Phi_x}{\partial y^{N-M}} + \frac{\partial^{M+1}\Phi}{\partial y^{M+1}}\frac{\partial^{N-M+1}\Phi}{\partial y^{N-M+1}}\right)\right\}\right], \quad (2.4A)$$

We can integrate the above system of equation (A.4) - (A.7) with respect to y and obtain the following integrals ;

for m = 0 :

$$\phi_y^{(n,0)} = \int_{-h_0}^{y} A^{(n,0)}\, dy \qquad \text{for } -h_0 \leq y \leq 0, \quad (A.9)$$

$$\eta^{(n,0)} = \frac{1}{g} C^{(n,0)} \qquad \text{at } y = 0, \quad (A.10)$$

and

$$B^{(n,0)} = \phi_y^{(n,0)} = \int_{-h_0}^{0} A^{(n,0)}\, dy, \qquad \text{at } y = 0, \quad (A.11)$$

for m ≠ 0 :

$$\phi^{(n,m)} = \frac{C_m}{C_{m0}}\psi^{(n,m)} + \frac{1}{mk_0}\left(S_m\int_{-h_0}^{y} A^{(n,m)} C_m\, dy - C_m\int_{-h_0}^{y} A^{(n,m)} S_m\, dy\right)$$
$$\text{for } -h_0 \leq y \leq 0, \quad (A.12)$$

$$g\eta^{(n,m)} = i m \omega_0 \psi^{(n,m)} + C^{(n,m)}, \qquad \text{at } y = 0, \quad (A.13)$$

and

$$(k_0 S_{m0} - m\omega_0 C_{m0}/g)\left(m\psi^{(n,m)} - \frac{1}{k_0}\int_{-h_0}^{0} A^{(n,m)} S_m\, dy\right) + (k_0 C_{m0} - m\omega_0 S_{m0}/g)\frac{1}{k_0}\int_{-h_0}^{0} A^{(n,m)} C_m\, dy$$
$$= B^{(n,m)} - i m \omega_0 C^{(n,m)}/g \qquad \text{at } y = 0, \quad (A.14)$$

where

$$C_m(y) = \cosh mk_0(y+h_0), \qquad S_m(y) = \sinh mk_0(y+h_0),$$
$$C_{m0} = C_m(0), \qquad S_{m0} = S_m(0), \quad (A.15)$$

and $\psi^{(n,m)}$ is a function of ξ and τ alone. Thus we can express $\phi^{(n,m)}$ and $\eta^{(n,m)}$ in terms of $\psi^{(1,1)}=\psi$ for (n, m) = (1,0), (1,1), (2,0), (2,1) and (2,2) :

$$\phi^{(1,1)}=\psi\cosh k(y+h_0)/C,\quad \phi_{\xi}^{(1,0)}=\beta_1|\psi|^2,$$

$$\phi^{(2,0)}=0,\ \phi^{(2,1)}=i\beta_2\psi_{\xi},\ \phi^{(2,2)}=i\beta_3\psi^2\cosh 2k(y+h_0)/C_{20} \tag{A.16}$$

$$\eta^{(1,0)}=0,\ g\eta^{(1,1)}=i\omega_0\psi,\quad g\eta^{(2,0)}=\gamma|\psi|^2,\ g\eta^{(2,1)}=V\psi_{\xi},\ g\eta^{(2,2)}=\gamma_2\psi^2,$$

with

$$\beta_1=\frac{V(\sigma^2-1)k_0^2-2\omega_0k_0}{gh_0-V^2},$$

$$\beta_2=-\frac{1}{k}\{k_0(y+h_0)\tanh k_0(y+h_0)-k_0h_0\sigma\}, \tag{A.17}$$

$$\beta_3=\frac{3k_0^2}{2\omega_0}\frac{\sigma C_{20}}{S^2}=\frac{3k_0^2(1-\sigma^4)}{4\omega_0\ \sigma^2},$$

where σ = tanh k_0h_0 , C = cosh k_0h_0 , S = sinh k_0h_0 C_{20} = cosh $2k_0h_0$, and γ and γ_2 are respectively given by (4.12) and (3.28).

These results are found to be in accordance with (4.11), (3.16), (3.21) and (4.4).

On the other hand, the consistency condition (A.8) for (n, m) = (3.1) gives the same equation for ψ as (4.5) :

$$i\frac{\partial\psi}{\partial\tau}+\mu\frac{\partial^2\psi}{\partial\xi^2}+\nu|\psi|^2\psi=0. \tag{A.18}$$

Appendix B. Explicit forms of $A^{(n,m)}$, $B^{(n,m)}$ and $C^{(n,m)}$

Explicit forms of $A^{(n,m)}$, $B^{(n,m)}$ and $C^{(n,m)}$ appeared in Appendix A are obtained from (2.1) - (2.4a) and (A.1) - (2.4A) as follows :

$$A^{(n,m)} = -2imk_0\phi_\xi^{(n-1,m)} - \phi_{\xi\xi}^{(n-2,m)}, \tag{B . 1}$$

$$\begin{aligned}
B^{(n,m)} =\ & \eta_\tau^{(n-2,m)} - V\eta_\xi^{(n-1,m)} - \langle \eta^{(n',m')}\phi_{yy}^{(n'',m'')}\rangle_{n,m} - \frac{1}{2}\langle \eta^{(n',m')}\eta^{(n'',m'')}\phi_{yyy}^{(n''',m''')}\rangle_{n,m} \\
& + \langle \eta_\xi^{(n',m')}\phi_\xi^{(n'',m'')}\rangle_{n-2,m} + ik_0\langle m'\eta^{(n',m')}\phi_\xi^{(n'',m'')}\rangle_{n-1,m} \\
& + ik_0\langle m''\eta_\xi^{(n',m')}\phi^{(n'',m'')}\rangle_{n-1,m} - k_0^2\langle m'm''\phi^{(n',m')}\eta^{(n'',m'')}\rangle_{n,m} \\
& + \langle \eta^{(n',m')}\eta_\xi^{(n'',m'')}\phi_{\xi y}^{(n''',m''')}\rangle_{n-2,m} + ik_0\langle m''\eta^{(n',m')}\eta_\xi^{(n'',m'')}\phi_y^{(n''',m''')}\rangle_{n-1,m} \\
& + ik_0\langle m''\eta^{(n',m')}\eta^{(n'',m'')}\phi_{\xi y}^{(n''',m''')}\rangle_{n-1,m} \\
& - k_0^2\langle m''m'''\eta^{(n',m')}\eta^{(n'',m'')}\phi_y^{(n''',m''')}\rangle_{n,m},
\end{aligned} \tag{B . 2}$$

$$\begin{aligned}
C^{(n,m)} =\ & -\phi_\tau^{(n-2,m)} + V\phi_\xi^{(n-1,m)} - \langle \eta^{(n',m')}\phi_{\tau y}^{(n'',m'')}\rangle_{n-2,m} \\
& + i\omega_0\langle m''\eta^{(n',m')}\phi_y^{(n'',m'')}\rangle_{n,m} + V\langle \eta^{(n',m')}\phi_{\xi y}^{(n'',m'')}\rangle_{n-1,m} \\
& - \frac{1}{2}\langle \eta^{(n',m')}\eta^{(n'',m'')}\phi_{\tau yy}^{(n''',m''')}\rangle_{n-2,m} + \frac{1}{2}i\omega_0\langle m'''\eta^{(n',m')}\eta^{(n'',m'')}\phi_{yy}^{(n''',m''')}\rangle_{n,m} \\
& + \frac{V}{2}\langle \eta^{(n',m')}\eta^{(n'',m'')}\phi_{\xi yy}^{(n''',m''')}\rangle_{n-1,m} - \frac{1}{2}\langle \phi_\xi^{(n',m')}\phi_\xi^{(n'',m'')}\rangle_{n-2,m} \\
& - \frac{1}{2}ik_0\langle m''\phi_\xi^{(n',m')}\phi^{(n'',m'')}\rangle_{n-1,m} - \frac{1}{2}ik_0\langle m'\phi^{(n',m')}\phi_\xi^{(n'',m'')}\rangle_{n-1,m} \\
& + \frac{1}{2}k_0^2\langle m'm''\phi^{(n',m')}\phi^{(n'',m'')}\rangle_{n,m} - \langle \eta^{(n',m')}\phi_\xi^{(n'',m'')}\phi_{\xi y}^{(n''',m''')}\rangle_{n-2,m} \\
& - ik_0\langle m'''\eta^{(n',m')}\phi_\xi^{(n'',m'')}\phi_y^{(n''',m''')}\rangle_{n-1,m} - ik_0\langle m''\phi^{(n',m')}\eta^{(n'',m'')}\phi_{\xi y}^{(n''',m''')}\rangle_{n-1,m} \\
& + k_0^2\langle m''m'''\eta^{(n',m')}\phi^{(n'',m'')}\phi_y^{(n''',m''')}\rangle_{n,m} - \frac{1}{2}\langle \phi_y^{(n',m')}\phi_y^{(n'',m'')}\rangle_{n,m} \\
& - \langle \eta^{(n',m')}\phi_y^{(n'',m'')}\phi_{yy}^{(n''',m''')}\rangle_{n,m},
\end{aligned} \tag{B . 3}$$

where the bracket $< \quad >_{n,m}$ denotes the coefficient of the m-th harmonics with n-th order with respect to ε , e.g.,

$$< m'' \phi_\xi^{(n',m')} \phi^{(n'',m'')} >_{n,m} = \sum_{n'+n''=n} \sum_{m'+m''=m} m'' \phi_\xi^{(n',m')} \phi^{(n'',m'')} ,$$

The consistency conditions (A.11) and (A.14) for (n, m) = (1,0) and (2,0) are trivially satisfied as easily seen from (B . 1) (B . 3) . Those for (n, m) = (1,1) and (2,1) lead to dispersion relation (3.17) and the reasonable foundation to take V as the group velocity. Whereas those for (n, m) = (3,0) and (2,2) determine $\phi_\xi^{(1,0)}$ and $\phi^{(2,2)}$ in terms of ψ . The condition for (n, m) = (3,1) requires that ψ is governed by the nonlinear Schrödinger equation (A.19).

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